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LETTER TO THE EDITOR

Quantum indistinguishability: alternative constructions of the transported basis

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Abstract. Our earlier arguments (Berry M V and Robbins J M 1997 *Proc. R. Soc. Lond.* A **453** 1771–90) leading to the spin–statistics relation are summarized and then revisited. Constructions are described that satisfy all our previous requirements but lead to the wrong exchange sign (one such alternative is the replacement of commutators by anticommutators in the Schwinger representation of the spins of the particles); we suggest why these might be unacceptable.

1. Introduction

Following our earlier exploration (Berry and Robbins 1997) (hereinafter called I) of the connection between spin and statistics for identical quantum particles in nonrelativistic quantum mechanics, and comments from several people, we wish to elaborate and extend our arguments. In this Letter we concentrate on one point: construction of the transported basis on which our scheme depends.

In section 2, we summarize the scheme for two particles with spin S (integer or half-integer), state the problem of the construction of the transported basis, and describe the construction based on Schwinger's representation of spin. In section 3 we present an argument, independent of the Schwinger representation, in which a correctly transported basis emerges from simple additional mathematical assumptions. But there exist alternative constructions that satisfy the conditions in I yet lead to the wrong spin–statistics connection; in section 4 we describe two of these. Section 5 gives some speculations about principles that might exclude constructions leading to the wrong statistics. Appendix A corrects a minor logical error in the derivation of the exchange sign in I, and appendix B gives technical details of one of the alternative constructions.

We should mention several related topics that we hope to discuss elsewhere but not here. First, the extension of our construction to $N > 2$ particles, where Atiyah (2000) has provided an explicit construction completing the programme outlined in section 6 of I, with far-reaching mathematical generalizations. Second, the relation between our nonrelativistic treatment and the more familiar arguments (Streater and Wightman 1964, Duck and Sudarshan 1997, 1998) based on relativistic quantum field theory (however, see Anandan (1998) for a relativistic extension of I). Third, the extension of our construction to include identical particles with additional properties such as isospin and colour.

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2. Reprise

In I, the state of the particles is represented by

$$|\Psi(\boldsymbol{r})\rangle = \sum_M \psi_M(\boldsymbol{r})\ |M(\boldsymbol{r})\rangle. \tag{1}$$

Here, $\boldsymbol{r} \equiv \boldsymbol{r}_1 - \boldsymbol{r}_2$ is the relative position vector for the particles (the dependence on the centre of mass is implicit); therefore exchange of positions corresponds to $\boldsymbol{r} \rightarrow -\boldsymbol{r}$. $M \equiv \{m_1, m_2\}$ labels the spin state of the particles, with m denoting the z component of spin; exchange of spins corresponds to $M \rightarrow \bar{M} \equiv \{m_2, m_1\}$. $|M(\boldsymbol{r})\rangle$ is the *transported spin basis*; that is, a basis for representing spins in a way that depends on the relative position of the particles. We require the transported basis to be single-valued, smooth, and also *parallel-transported*; that is,

$$A_{M,M'} \equiv \mathrm{i}\left\langle M'(\boldsymbol{r})\middle|\nabla M\ (\boldsymbol{r})\right\rangle = 0. \tag{2}$$

By introducing $|M(\boldsymbol{r})\rangle$, we can incorporate the indistinguishability of the particles by identifying $\boldsymbol{r}$ and $-\boldsymbol{r}$. For this, it is necessary to exchange the spins along with the positions, leading to the exchange requirement

$$|M(-\boldsymbol{r})\rangle = (-1)^K \left|\bar{M}(\boldsymbol{r})\right\rangle \tag{3}$$

involving the exchange sign $(-1)^K$ (K integer). Parallel transport excludes the possibility of a more general exchange phase, depending on $\boldsymbol{r}$ (see appendix A): the phase is a multiple of π, ensuring that the physics associated with it does not depend on the relative positions of the particles.

The basis is a set of $(2S+1)^2$ spinors; each is a single-valued function of $\boldsymbol{r}$. The basis inhabits a larger ambient space, within which it is smoothly transported. This enlargement is a necessary consequence of the parallel transport requirement; without enlargement, (2) would imply that $|M(\boldsymbol{r})\rangle$ is independent of $\boldsymbol{r}$ and so unable to satisfy the fundamental exchange requirement (3). For further discussion of this requirement, and of the augmented spin space, see Robbins (2000).

With the basis $|M(\boldsymbol{r})\rangle$, the inversion $\boldsymbol{r} \rightarrow -\boldsymbol{r}$ corresponds to complete exchange of the particles, motivating the central step, which is to impose single-valuedness on $|\Psi(\boldsymbol{r})\rangle$, regarded as a function taking values in the Hilbert space of the two spins, and whose domain is the product of the projective plane (sphere with identified antipodes) with the radial coordinate $|\boldsymbol{r}|$. Thus

$$|\Psi(\boldsymbol{r})\rangle = |\Psi\ (-\boldsymbol{r})\rangle. \tag{4}$$

With (1) and (3), single-valuedness implies that the exchange phase acquired by the basis $|M(\boldsymbol{r})\rangle$ is inherited by the coefficients $\psi_M(\boldsymbol{r})$; that is,

$$\psi_{\bar{M}}(-\boldsymbol{r}) = (-1)^K\ \psi_M(\boldsymbol{r}). \tag{5}$$

We emphasize that this is a natural extension to two identical particles of the requirement of single-valuedness of wavefunctions familiar in one-body quantum mechanics: because of the way we define the configuration space, $\boldsymbol{r}$ and $-\boldsymbol{r}$ are the same point. (Some other nonrelativistic accounts of the spin–statistics relation (Broyles 1976, Bacry 1995) also invoke notions of single-valuedness; these and other treatments are discussed in comprehensive reviews (Duck and Sudarshan 1997, 1998) of previous studies of the spin–statistics relation.)

In I, the coefficients $\psi_M(\boldsymbol{r})$ were shown to be the same as those in the conventional formulation, in which the state is represented in terms of the fixed-spin basis states $|M\rangle$; that is,

$$|\Psi_{\text{fixed}}(\boldsymbol{r})\rangle = \sum_M \psi_M(\boldsymbol{r})\,|M\rangle. \tag{6}$$

Therefore equation (5) is a generalized spin–statistics relation, with exchange sign $(-1)^K$ originating in the behaviour of the transported basis under exchange. Obviously, the central problem in our formalism is the determination of this sign.

In section 4 of I an explicit construction was given for the transported basis, leading to the exchange sign

$$K = 2S; \tag{7}$$

that is,

$$\left|\bar{M}(-\boldsymbol{r})\right\rangle = (-1)^{2S}\,|M(\boldsymbol{r})\rangle \tag{8}$$

so that (5) becomes

$$\psi_{\bar{M}}(-\boldsymbol{r}) = (-1)^{2S}\,\psi_M(\boldsymbol{r}). \tag{9}$$

This is the correct form of the spin–statistics relation.

The construction of the basis in I made use of the Schwinger representation (Schwinger 1965, Sakurai 1994). In this, each of the spins 1 and 2 is regarded as made from two harmonic oscillators, a_1, b_1 for the spin 1 and a_2, b_2 for the spin 2, with associated annihilation and creation operators $\mathbf{a}_1, \mathbf{a}_1^\dagger$, etc; operators corresponding to different oscillators commute. As parallel transport requires, the resulting transported basis $|M(\boldsymbol{r})\rangle$ inhabits an augmented space of spins, with dimension $d = (4S+1)(4S+2)(4S+3)/6$ (the number of ways that $4S$ quanta can be distributed among four oscillators) that is larger than the dimension $(2S+1)^2$ of the fixed-spin space. We can state this another way, in terms of the unitary operator $\mathbf{U}(\boldsymbol{r})$ generating the transported basis from the fixed basis by

$$|M(\boldsymbol{r})\rangle = \mathbf{U}(\boldsymbol{r})\,|M\rangle \tag{10}$$

as follows: the $(2S+1)^2$ vectors $|M\rangle$ do not span the space of states on which $\mathbf{U}(\boldsymbol{r})$ acts.

We argued (at the end of section 7 of I) that this enlargement is physically natural, even though the expansion of $|M(\boldsymbol{r})\rangle$ seems to involve fixed-basis states where the two particles would have different spins. Indeed, the formalism guarantees that these unphysical values of spin are never realized. The reason is that in the transported basis the two spins must be represented not by the fixed matrices $\mathbf{S}_1, \mathbf{S}_2$ but by the transported operators

$$\begin{aligned}\mathbf{S}_1(\boldsymbol{r}) &= \mathbf{U}(\boldsymbol{r})\mathbf{S}_1\mathbf{U}^\dagger(\boldsymbol{r})\\ \mathbf{S}_2(\boldsymbol{r}) &= \mathbf{U}(\boldsymbol{r})\mathbf{S}_2\mathbf{U}^\dagger(\boldsymbol{r})\end{aligned} \tag{11}$$

whose squared magnitudes always have the physically correct eigenvalue $S(S+1)$.

Let us exhibit this enlargement explicitly for two spin-$\frac{1}{2}$ particles, by writing the four transported basis states in terms of the ten states in the augmented spin space. The first four states are the original fixed-basis states of the two particles, which in an obvious notation are

$$|+\,+\rangle\,,\quad |+\,-\rangle\,,\quad |-\,+\rangle\,,\quad |-\,-\rangle\,. \tag{12}$$

The remaining six correspond to fixed-spin states in which one spin is zero and the other spin is one; that is, using the notation $|S_1, S_2; m_1, m_2\rangle$,

$$\begin{aligned}&|0,1;0,1\rangle\,,\quad |0,1;0,0\rangle\,,\quad |0,1;0,-1\rangle\\ &|1,0;1,0\rangle\,,\quad |1,0;0,0\rangle\,,\quad |1,0;-1,0\rangle\,.\end{aligned} \tag{13}$$

Then the transported states, as functions of the polar angles θ, ϕ of $\boldsymbol{r}$, are

$$
\begin{aligned}
|+\,+\,(\boldsymbol{r})\rangle &= \tfrac{1}{\sqrt{2}}\sin\theta\,(\exp\{\mathrm{i}\phi\}\,|0,1;0,1\rangle - \exp\{-\mathrm{i}\phi\}\,|1,0;1,0\rangle) + \cos\theta\,|++\rangle \\
|+\,-\,(\boldsymbol{r})\rangle &= \tfrac{1}{2}\left[\sin\theta\,(-\exp\{-\mathrm{i}\phi\}\,|1,0;0,0\rangle + \exp\{\mathrm{i}\phi\}\,|0,1;0,0\rangle)\right] \\
&\quad + \tfrac{1}{2}\,[(\cos\theta+1)\,|+-\rangle + (\cos\theta-1)\,|-+\rangle] \\
|-\,+\,(\boldsymbol{r})\rangle &= \tfrac{1}{2}\left[\sin\theta\,(-\exp\{-\mathrm{i}\phi\}\,|1,0;0,0\rangle + \exp\{\mathrm{i}\phi\}\,|0,1;0,0\rangle)\right] \\
&\quad + \tfrac{1}{2}\,[(\cos\theta-1)\,|+-\rangle + (\cos\theta+1)\,|-+\rangle] \\
|-\,-\,(\boldsymbol{r})\rangle &= \tfrac{1}{\sqrt{2}}\sin\theta\,(\exp\{\mathrm{i}\phi\}\,|0,1;0,-1\rangle - \exp\{-\mathrm{i}\phi\}\,|1,0;-1,0\rangle) + \cos\theta|--\rangle.
\end{aligned}
\tag{14}
$$

It is easy to confirm that these states are orthonormal, and single-valued and smooth functions of $\boldsymbol{r}$. They also satisfy the parallel-transport requirement (2): the derivatives of each state with respect to θ and ϕ are orthogonal to all four states.

At the end of I we conjectured that any single-valued, smooth and parallel-transported basis would yield the correct exchange phase (6). That is wrong, as we will demonstrate in section 4.

3. Supplementary assumptions yielding the correct sign

There is a simple mathematical argument, independent of the Schwinger construction, that yields the correct exchange sign, at least for two particles. The argument is based on the following two additional assumptions. (i) There are no degeneracies in the augmented space of spins; that is, each combination of S_1, S_2, m_1, m_2 is represented by at most one state among those spanning the space acted on by $\mathbf{U}(\boldsymbol{r})$. (ii) In the augmented space of spins, the magnitude of the total spin is conserved as $\boldsymbol{r}$ varies.

These assumptions imply the exchange sign $(-1)^{2S}$. To show this, we represent the transported basis using not the z components $M \equiv \{m_1, m_2\}$ of the separate spins but the magnitude j and the z component j_z of the total spin $\mathbf{S}_1 + \mathbf{S}_2$. Defining $J \equiv \{j, j_z\}$, we denote states in this representation by $|J\rangle$, and the corresponding transported states by $|J(\boldsymbol{r})\rangle$. From the Clebsch–Gordan coefficients, or otherwise, it is possible to show that under spin exchange $|M\rangle \to |\bar{M}\rangle$ the states $|J\rangle$ transform as

$$|J\rangle \to (-1)^{2S-j}\,|J\rangle. \tag{15}$$

Then the exchange rule (3) for the transported basis, which can also be written in the form

$$\left|\bar{M}(-\boldsymbol{r})\right\rangle = (-1)^K\,|M(\boldsymbol{r})\rangle, \tag{16}$$

becomes

$$|J(-\boldsymbol{r})\rangle = (-1)^{K+2S-j}\,|J(\boldsymbol{r})\rangle. \tag{17}$$

To determine the exchange sign K, we need consider only one of the states J, since K is the same for all of them. We choose the singlet state $J = \{j = 0, j_z = 0\}$. By assumption (i), there is only one such state in the basis of fixed states spanning $\mathbf{U}(\boldsymbol{r})$. By assumption (ii), this state remains isolated when transported, and so can be written in the simple form

$$|\{j = 0, j_z = 0\}\,(\boldsymbol{r})\rangle = \exp\{\mathrm{i}\delta(\boldsymbol{r})\}\,|\{j = 0, j_z = 0\}\rangle. \tag{18}$$

Parallel transport (equation (2)) now gives $\delta(\boldsymbol{r}) = 0$ immediately.

Therefore the singlet is invariant under transport and therefore also under exchange. Applying this invariance to (17) with $j = 0$ gives the correct sign $K = 2S$.

4. Alternative constructions yielding the wrong sign

There are constructions that satisfy the conditions in I but yield the wrong exchange sign. Here we describe two illustrative examples.

The first such construction applies to spin-zero particles. It is very simple: the single transported state is represented as the unit vector

$$|M(\boldsymbol{r})\rangle = |\{0,0\}(\boldsymbol{r})\rangle = \boldsymbol{r}/|\boldsymbol{r}|. \tag{19}$$

This is single-valued, smooth, and parallel-transported, and involves the extended spin space spanned by the three basis states $\boldsymbol{e}_x$, $\boldsymbol{e}_y$ and $\boldsymbol{e}_z$, of which only one (e.g. $\boldsymbol{e}_z$) corresponds to the fixed-spin state $|\{0,0\}\rangle$. The operator $\mathbf{U}(\boldsymbol{r})$ is then rotation from $\boldsymbol{e}_z$ to $\boldsymbol{r}$. Under $\boldsymbol{r} \to -\boldsymbol{r}$, $|M(\boldsymbol{r})\rangle$ changes sign fermionically, rather than being bosonically invariant.

In the second construction, we modify the Schwinger construction (section 4 of I) by replacing all commutators involving the operators $\mathbf{a}_1, \mathbf{a}_1^\dagger, \mathbf{b}_1, \mathbf{b}_1^\dagger, \mathbf{a}_2, \mathbf{a}_2^\dagger, \mathbf{b}_2, \mathbf{b}_2^\dagger$ by anticommutators. Because anticommutation implies $\mathbf{a}_1^2 = 0$, etc, this particular modification works only for spin-$\frac{1}{2}$ particles. After some calculation (outlined in appendix B) we find the following 'anti-Schwinger' transported basis, which should be compared with the Schwinger-generated basis (14):

$$\begin{aligned}
&|++(\boldsymbol{r})\rangle = |++\rangle \qquad |--(\boldsymbol{r})\rangle = |--\rangle \\
|+-(\boldsymbol{r})\rangle &= \tfrac{1}{2}\sin\theta\,(-\exp\{\mathrm{i}\phi\}\,|0,0;0,0\rangle_1 + \exp\{-\mathrm{i}\phi\}\,|0,0;0,0\rangle_2) \\
&\quad + \cos^2\tfrac{1}{2}\theta\,|+-\rangle + \sin^2\tfrac{1}{2}\theta\,|-+\rangle \\
|-+(\boldsymbol{r})\rangle &= \tfrac{1}{2}\sin\theta\,(\exp\{\mathrm{i}\phi\}\,|0,0;0,0\rangle_1 - \exp\{-\mathrm{i}\phi\}\,|0,0;0,0\rangle_2) \\
&\quad + \sin^2\tfrac{1}{2}\theta\,|+-\rangle + \cos^2\tfrac{1}{2}\theta\,|-+\rangle
\end{aligned} \tag{20}$$

where $|0,0;0,0\rangle_1$ and $|0,0;0,0\rangle_2$ are degenerate states where both particles have (fixed) spin zero.

These states are single-valued, smooth, and parallel-transported, and they satisfy the fundamental requirement that position exchange ($\boldsymbol{r} \to -\boldsymbol{r}$) is equivalent to spin exchange ($|M\rangle \to |\bar{M}\rangle$). But the sign is wrong: a plus (i.e. the bosonic exchange rule $|M(-\boldsymbol{r})\rangle = +|\bar{M}(\boldsymbol{r})\rangle$) instead of the fermionic minus.

5. Discussion

Clearly, the existence of alternative constructions means that the arguments in I, leading to the correct spin–statistics relation, cannot be regarded as a derivation from first principles. Those arguments would constitute a derivation if the implementation of exchange by the Schwinger-constructed transported basis were incorporated as an assumption, but the dependence on that particular formalism does not seem fundamental. We do not know a set of simple general assumptions that imply the correct connection. But all alternative constructions that we have found so far, that lead to the wrong spin–statistics connection, are unnatural or unsatisfactory in one way or another, and we offer the following 'exclusion principles' as worth considering in the search for a convincing general argument.

(a) Constructions should work for all S. Why? Because our arguments here relate to quantum physics, not elementary particle physics, and although quantum mechanics is a fundamental theory, its application (like that of Newtonian physics) is not restricted to fundamental particles. In particular, it can be applied to identical composites (e.g. atoms and α-particles), whose statistics must be those calculated from their constituents, which may combine to give any S. The Schwinger construction naturally includes all S, and also any number of particles

N. On the other hand, anti-Schwinger (equation (20)) fails this 'compositeness' test, because the restriction to spin-$\frac{1}{2}$ makes it impossible to build up such composites. (It is possible to generalize anti-Schwinger by representing each spin S in terms of $2S$ pairs of annihilation and creation operators (Georgi 1982); however, this has the disagreeable feature of introducing a different structure group SU($4SN$) for each S.)

(b) Constructions should be indecomposable: it should not be possible to express $|M(\boldsymbol{r})\rangle$ as a tensor product $|M_\alpha(\boldsymbol{r})\rangle \otimes |M_\beta(\boldsymbol{r})\rangle$, where $|M_\alpha(\boldsymbol{r})\rangle$ satisfies the properties required of the transported basis (exchange under $\boldsymbol{r} \to -\boldsymbol{r}$, smoothness, parallel transport), unless the second factor $|M_\beta(\boldsymbol{r})\rangle$ is a constant vector (that is, independent of M and $\boldsymbol{r}$). The Schwinger construction is indecomposable in this sense, but (19) (and generalizations thereof) are not.

(c) Constructions should be intrinsically related to spin. This excludes (19) (and generalizations thereof), where the augmented spin space contains states arbitrarily appended to those representing spins. The Schwinger construction satisfies this condition, because it is based on a representation of spin.

(d) The physical hypothesis can be made that quantum spins are built from Schwinger's oscillators. The spin–statistics connection follows from this hypothesis, by a slight rephrasing of the arguments in I. But for this to be convincing there should be some other consequence, in addition to spin–statistics, of the existence of these 'atomic spin bosons'.

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Appendix A. Effect of double exchange

The most general exchange involving a phase is

$$|M(-\boldsymbol{r})\rangle = \exp\{i\gamma(\boldsymbol{r})\}\left|\bar{M}(\boldsymbol{r})\right\rangle \tag{A.1}$$

(the further generalization, to a phase depending on the state M, is excluded by the obvious requirement that the exchange is preserved for superpositions of the states M, e.g. those corresponding to rotations of the quantization direction).

In section 2 of I we argued that single-valuedness of the transported basis, embodied in a double application of $\boldsymbol{r} \to -\boldsymbol{r}$, requires the exchange phase factor to be a sign; that is, $\exp\{i\gamma(\boldsymbol{r})\} = (-1)^K$. This was wrong. In fact, single-valuedness under double exchange requires

$$\begin{aligned} |M(\boldsymbol{r})\rangle &= |M(-(-\boldsymbol{r}))\rangle \\ &= \exp\{i\gamma(-\boldsymbol{r})\}\left|\bar{M}(-\boldsymbol{r})\right\rangle \\ &= \exp\{i[\gamma(-\boldsymbol{r}) + \gamma(\boldsymbol{r})]\}\left|M(\boldsymbol{r})\right\rangle \end{aligned} \tag{A.2}$$

implying

$$\gamma(\boldsymbol{r}) = \pi K + \mu(\boldsymbol{r}) \tag{A.3}$$

where K is an integer and $\mu(-\boldsymbol{r}) = -\mu(\boldsymbol{r})$.

The exchange rule generated by (A.3) is

$$|M(-\boldsymbol{r})\rangle = (-1)^K \exp\{i\mu(\boldsymbol{r})\}\,|\bar{M}(\boldsymbol{r})\rangle. \tag{A.4}$$

However, the parallel-transport requirement (2) ensures that $\mu(r)$ vanishes. This is because

$$A_{M,M'}(-r) = A_{\bar{M},\bar{M}'}(r) - \nabla\mu(r)\delta_{M,M'} \qquad \text{(A.5)}$$

so that satisfying (2) for all r, M, M' requires that μ be a constant, which must vanish since the function $\mu(r)$ is odd. Thus the exchange phase factor is indeed a sign as in (3), and the resulting physics (spin–statistics relation) is isotropic; that is, independent of the relative position of the particles, as it must be.

We emphasize that parallel transport (as well as the exchange sign (7)) is a consequence of the Schwinger construction of I; it does not have to be imposed separately.

Appendix B. Anti-Schwinger construction

The logic of this construction follows closely that of section 4 of I: in terms of the creation and annihilation operators $\mathbf{a}_1, \mathbf{a}_1^\dagger, \mathbf{b}_1, \mathbf{b}_1^\dagger, \mathbf{a}_2, \mathbf{a}_2^\dagger, \mathbf{b}_2, \mathbf{b}_2^\dagger$ for the four oscillators, spins are defined by (4.1) of I, fixed-basis states are defined in terms of the vacuum state of the oscillators by (4.11) of I, exchange angular momentum $\mathbf{E} = \mathbf{E}_a + \mathbf{E}_b$ by (4.4) and (4.5) of I, transport by the exchange rotation operator $\mathbf{U}(r)$ (4.10) of I, and its action on the fixed basis by (4.19) of I.

The crucial difference is that now the operators anticommute. Thus

$$\begin{aligned} &\mathbf{a}_1\mathbf{a}_1^\dagger + \mathbf{a}_1^\dagger\mathbf{a}_1 = 1, \quad \text{etc} \\ &\mathbf{a}_1\mathbf{b}_1 = -\mathbf{b}_1\mathbf{a}_1, \quad \text{etc} \\ &\mathbf{a}_1^2 = 0, \quad \text{etc.} \end{aligned} \qquad \text{(B.1)}$$

In particular, the last equalities imply that each oscillator can be occupied by at most one quantum. In terms of the vacuum state $|\mathbf{0}\rangle$ (where all four oscillators are unoccupied) the first four fixed-basis states are

$$\begin{aligned} |+\,+\rangle &= \mathbf{a}_1^\dagger\mathbf{a}_2^\dagger\,|\mathbf{0}\rangle \\ |+\,-\rangle &= \mathbf{a}_1^\dagger\mathbf{b}_2^\dagger\,|\mathbf{0}\rangle \\ |-\,+\rangle &= \mathbf{b}_1^\dagger\mathbf{a}_2^\dagger\,|\mathbf{0}\rangle \\ |-\,-\rangle &= \mathbf{b}_1^\dagger\mathbf{b}_2^\dagger|\mathbf{0}\rangle. \end{aligned} \qquad \text{(B.2)}$$

Each + or − corresponds to a spin-half (up or down).

There cannot be any fixed-spin-one states, as in the six additional Schwinger basis states, because these would correspond to oscillators occupied by more than one quantum. Instead, there are only two more basis states, namely, in the notation $|S_1, S_2; m_1, m_2\rangle$,

$$\begin{aligned} |0, 0; 0, 0\rangle_1 &= \mathbf{a}_1^\dagger\mathbf{b}_1^\dagger\,|\mathbf{0}\rangle \\ |0, 0; 0, 0\rangle_2 &= \mathbf{a}_2^\dagger\mathbf{b}_2^\dagger|\mathbf{0}\rangle. \end{aligned} \qquad \text{(B.3)}$$

The fact that these states correspond to spin-zero, rather than spin-one, is a consequence of the anticommutation relations, which imply the following formula for the operator for the magnitude of a single spin:

$$S \cdot S = \tfrac{3}{4}\left(\mathbf{a}^\dagger\mathbf{a} + \mathbf{b}^\dagger\mathbf{b} - 2\mathbf{a}^\dagger\mathbf{a}\mathbf{b}^\dagger\mathbf{b}\right). \qquad \text{(B.4)}$$

Calculation of the action of the exchange rotation operator $\mathbf{U}(r)$ depends on the fact that the operators $\mathbf{E}_a$ and $\mathbf{E}_b$ commute, even though the individual $\mathbf{a}$ and $\mathbf{b}$ operators do not, and the relation (4.19) of I remains valid in the anti-Schwinger construction. With these observations, the derivation of the anti-Schwinger transported basis (20) is straightforward.

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